

MANIFESTLY GAUGE INVARIANT COMPUTATIONS

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Submitted May 14, 2002

Using a gauge invariant exact renormalization group, we show how to compute the effective action, and extract the physics, whilst manifestly preserving gauge invariance at each and every step. As an example we give an elegant computation of the one-loop $SU(N)$ Yang-Mills beta function, for the first time at finite N without any gauge fixing or ghosts. It is also completely independent of the details put in by hand, *e.g.* the choice of covariantisation and the cutoff profile, and, therefore, guides us to a procedure for streamlined calculations.

PACS: 11.10.Hi, 11.10.Gh, 11.15.Tk

1 Introduction

The continuum formulation of the Wilsonian Renormalization Group (RG) [1], *a.k.a.* the Exact Renormalization Group, has the potential to be an extremely powerful framework for both exact and approximate calculations in non-perturbative quantum field theory [2].

However, a crucial challenge to confront has existed since then: the application to continuum gauge field theories. Since the introduction of an effective cutoff breaks the gauge invariance, the Wilsonian approach cannot be naïvely followed.

Following the earlier works of one of us [3], we have proposed a manifestly gauge invariant exact RG equation, by which we can compute the effective action and extract the physics, without ever requiring any gauge fixing or ghosts. The Polchinski equation [4] is modified by *e.g.* introducing a Wilson line between functional derivatives, and thus it becomes gauge invariant.

Needless to say, the theory is also to be regularised in a gauge invariant way, and this is why a Poincaré and gauge invariant regularisation scheme has been worked out that is based on a real ultraviolet scale.⁴ It works for pure $SU(N)$ Yang-Mills theory in dimension four or less and, in the large N limit, in any dimension [5, 6].

Such a regularisation scheme can be very nicely incorporated into our gauge invariant formulation of the RG and, together, they constitute a very powerful tool for investigating non-perturbative aspects of quantum theories. Indeed, the use of manifest gauge invariance is the key ingredient throughout the calculation.

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⁴as opposed to *e.g.* analytic continuation of perturbative amplitudes in dimensional regularisation.

The relations (2) differ from the usual Polchinski-type equation by the presence of terms involving commutators.⁶ These latter, once the theory is spontaneously broken, will affect the equations for the two-point bosonic and fermionic vertices differently, and by choosing the related kernels properly, all the two-point vertices of the effective action can be set equal to the hatted ones (cf. [6, 8]). This last requirement will greatly simplify the equations for the higher order interactions, and thus will be enforced for convenience.

The resulting kernels in eq. (2) are found to be

$$M(x) = - \left(\frac{2c^2}{x\tilde{c} + 2c} \right)', \quad xH(x) = \left(\frac{2x^2\tilde{c}}{x + 2\lambda\tilde{c}} \right)', \quad xL(x) = \left(\frac{x^2\tilde{c}(\lambda\tilde{c}^2 - c)}{(x + 2\lambda\tilde{c})(x\tilde{c} + 2c)} \right)', \quad (3)$$

where prime denotes differentiation with respect to x and c, \tilde{c} are meant to be functions of x .

Breaking supersymmetry spontaneously in the fermionic directions causes the fermionic fields to acquire a mass of order the effective cutoff, whilst leaving the physical gauge field and its copy massless. In the unitary gauge, the Goldstone modes D vanish (eaten by the B 's), leaving the massive ‘‘Higgs’’ C and the massive vector fermions, B_μ . The symmetries thus suggest that superfields be split into their diagonal and off-diagonal components, *i.e.* $\mathcal{A}_\mu = A_\mu + B_\mu$ and $\mathcal{C} = C + D$ [6, 8]. This makes the equations much easier to derive and, also, resembles what is usually done in the context of the standard model, namely to deal with massive and massless combination of gauge fields rather than with the fields themselves. The kernels and their expansion in powers of gauge fields may be split accordingly. (See Figs. 2 and 3 for some of those actually used in the calculation.)

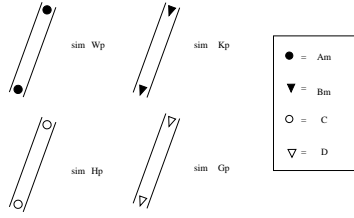


Fig. 2. Graphical representation of 0-point kernels. The f -kernel in Fig. 1 stands for any of these. $K \doteq c' + M$.

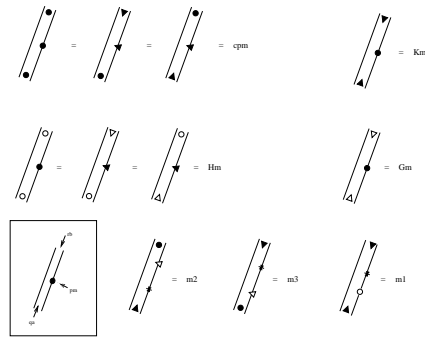


Fig. 3. Graphical representation of 1-point kernels. The boxed diagram indicates the position of incoming momenta. The σ_3 s coming from the symmetry breaking are represented by stars, while (“) stands for $(p; q, r)$.

Expanding the action and the beta function $\beta(g) = \Lambda \partial_\Lambda g$ in powers of the coupling constant:

$$S = \frac{1}{g^2} S_0 + S_1 + g^2 S_2 + \dots \quad \beta = \Lambda \partial_\Lambda g = \beta_1 g^3 + \beta_2 g^5 + \dots \quad (4)$$

⁶The commutators guarantee the equation is still gauge invariant.

yields the loopwise expansion of the flow equation

$$\Lambda \partial_\Lambda S_0 = -a_0[S_0, S_0 - 2\hat{S}], \quad (5)$$

$$\Lambda \partial_\Lambda S_1 = 2\beta_1 S_0 - 2a_0[S_0 - \hat{S}, S_1] + a_1[S_0 - 2\hat{S}], \quad (6)$$

etc., where S_0 (S_1) is the classical (one-loop) effective action. The one-loop coefficient, β_1 , can be extracted directly from eq. (6) once the renormalization conditions are imposed. Since gauge invariance already forces the anomalous dimension of the gauge field to vanish [3, 6, 8], we only need to define the renormalized coupling $g(\Lambda)$. This is done via the field strength for the physical gauge field, A_μ^1 [3, 6, 8],

$$S = \frac{1}{2g^2(\Lambda)} \text{tr} \int d^D x (F_{\mu\nu}^1)^2 + \mathcal{O}(\nabla^3). \quad (7)$$

From the above equation

$$S_{\mu\nu}^{AA}(p) + S_{\mu\nu}^{AA\sigma}(p) = \frac{2}{g^2} \square_{\mu\nu}(p) + \mathcal{O}(p^3) = \frac{1}{g^2} S_0^{AA}(p) + \mathcal{O}(p^3), \quad (8)$$

with $\square_{\mu\nu}(p)$ being the transverse combination ($p^2 \delta_{\mu\nu} - p_\mu p_\nu$). Eq. (8) implies the $\mathcal{O}(p^2)$ component of all the higher loop contributions $S_{n\mu\nu}^{AA}(p) + S_{n\mu\nu}^{AA\sigma}(p)$ must vanish. Thus the equation for β_1 becomes algebraic ($\Sigma_0 = S_0 - 2\hat{S}$):

$$-2\beta_1 S_0^{AA}(p) + \mathcal{O}(p^3) = a_1[\Sigma_0]_{\mu\nu}^{AA}(p). \quad (9)$$

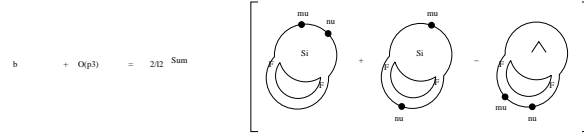


Fig. 4. Graphical representation of eq. (9).

In order to calculate the r.h.s. of eq. (9), we will adopt the following strategy:

- i. introduce the “integrated kernels” in the S_0 part of the first diagram and integrate by parts so as to end up with Λ -derivatives of vertices of the effective action;
- ii. use the equations of motion for the effective couplings;
- iii. use the relation between the integrated kernels and their corresponding two-point functions to simplify the diagrams obtained so far;
- iv. repeat the above procedure when any three-point effective coupling is generated.

This simple procedure, which will be described in more detail in the next section, ensures that any dependence upon n -point vertices of the seed action, $n \geq 3$, will cancel out. This implies that the calculation is actually independent of the choice of \hat{S} , provided it is a covariantisation of its two-point vertices⁷ and these latter vertices are infinitely differentiable and lead to convergent momentum integrals [9, 10, 6]. Moreover, pursuing that strategy will also guarantee that just the kernels’ vertices with special momenta remain that by gauge invariance can be expressed as derivatives of their generators (for an example see Section 3), which means independence of the choice of covariantisation.

⁷set equal to the effective ones for convenience.

3 (Un-)Broken gauge invariance

The invariance under the (broken) $SU(N|N)$ gauge symmetry results in the following set of trivial Ward identities

$$\begin{aligned} q^\nu U_{\dots a \nu b \dots}^{XY \dots}(\dots p, q, r, \dots) &= U_{\dots a b \dots}^{XY \dots}(\dots p, q + r, \dots) - U_{\dots a b \dots}^{XY \dots}(\dots p + q, r, \dots), \\ q^\nu U_{\dots a \nu b \dots}^{BY \dots}(\dots p, q, r, \dots) &= \pm U_{\dots a b \dots}^{X\hat{Y} \dots}(\dots p, q + r, \dots) \mp U_{\dots a b \dots}^{X\hat{Y} \dots}(\dots p + q, r, \dots) \\ &\quad + 2U_{\dots a b \dots}^{D\sigma Y \dots}(\dots p, q, r, \dots), \end{aligned} \quad (10)$$

where U is any vertex, a and b are Lorentz indices or null as appropriate and \hat{X}, \hat{Y} are opposite statistics partners of X, Y . The sign of the terms containing \hat{X}, \hat{Y} depends on whether B goes past a σ_3 on its way back and forth.

By specialising (10) to a proper set of momenta, one of which has to be infinitesimal, it is possible to express n -point vertices with one null momentum as derivatives of $(n - 1)$ -point's, independently of the choice of covariantisation. As an example, let us consider the three-point pure- A effective vertex at vanishing first momentum, $S_{\mu\nu\rho}^{AAA}(0, k, -k)$. By using (10),

$$\epsilon^\mu S_{\mu\nu\rho}^{AAA}(\epsilon, k, -k - \epsilon) = S_{\nu\rho}^{AA}(k + \epsilon) - S_{\nu\rho}^{AA}(k) = \epsilon^\mu \partial_\mu S_{\nu\rho}^{AA}(k) + \mathcal{O}(\epsilon^2). \quad (11)$$

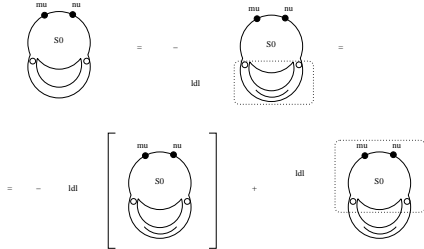
At order ϵ , $S_{\mu\nu\rho}^{AAA}(0, k, -k) = \partial_\mu^k S_{\nu\rho}^{AA}(k)$. Also $S_{\mu\nu\rho\sigma}^{AAAA}(0, 0, k, -k) = 12\partial_\mu^k \partial_\nu^k S_{\rho\sigma}^{AA}(k)$.

4 A sample of the calculation: the C sector

In this section the simplest part of the computation will be described, that is the scalar sector. All the steps of the strategy previously outlined will be illustrated by means of diagrams, as the cancellations taking place are evident already at that level. Of course, performing the full and complete calculation yields the same result.

We start by defining the integrated kernel. As $\Lambda \partial_\Lambda f(\frac{p^2}{\Lambda^2}) = -2\frac{p^2}{\Lambda^2} f'(\frac{p^2}{\Lambda^2})$,

$$\frac{1}{\Lambda^4} H = -\frac{1}{2p^4} \Lambda \partial_\Lambda \left(\frac{2x^2 \tilde{c}}{x + 2\lambda \tilde{c}} \right) = -\Lambda \partial_\Lambda \Delta^{CC} \quad (12) \quad \left| \begin{array}{c} \circ \\ \circ \end{array} \right| = -\text{Idl} \quad \left| \begin{array}{c} \circ \\ | \\ \circ \end{array} \right|$$



The integrated kernel is introduced via eq. (12) into the S_0 part of the first diagram in Fig. 4. One then integrates by parts, so as to end up with a total Λ -derivative plus the tree-level $\Lambda \partial_\Lambda S_{\mu\nu}^{AAC C}$ vertex joined by a Δ^{CC} . The latter will be dealt with, using the equations of motion.

The next step consists in using eq. (5) as specialised to $S_{\mu\nu}^{AAC C}$. Some of the diagrams are shown in Fig. 5.

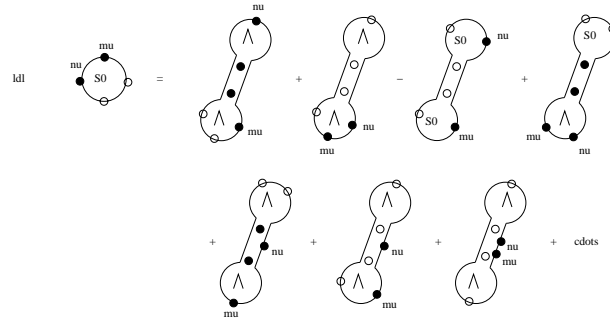
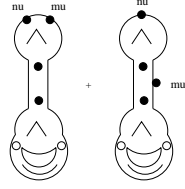


Fig. 5. Eq. (5) as specialised to $S_{\mu\nu}^{AACC}$. The ellipsis stands for similar diagrams which have not been drawn.

Already at this level, we note that some of the diagrams either do not contribute at all (cf. Fig. 6) or they give a potentially universal contribution, *i.e.* something depending only on two-point vertices and integrated kernels (cf. Fig. 7).



$$\left(\hat{S}_{\mu\nu\alpha}^{AAA}(p, -p, 0) c'(0) \hat{S}_{\alpha}^{ACC}(0, k, -k) + \hat{S}_{\nu\alpha}^{AA}(p) c'_{\mu}(p; 0, -p) \hat{S}_{\alpha}^{ACC}(0, k, -k) \right) \Delta^{CC}(k) \Big|_{p^2} = 0$$

Fig. 6. Diagrams not contributing to β_1 .



$$\begin{aligned} & \hat{S}_{\nu\alpha}^{AA}(p) c'(p^2 \Lambda^2) \hat{S}_{\mu\alpha}^{AACC}(p, -p, k, -k) \Delta^{CC}(k) \Big|_{p^2} = \\ & \hat{S}_{\nu\alpha}^{AA}(p) c'(0) \hat{S}_{\mu\alpha}^{AACC}(0, 0, k, -k) \Delta^{CC}(k) = \\ & 12 \hat{S}_{\nu\alpha}^{AA}(p) c'(0) \partial_{\mu}^k \partial_{\alpha}^k \hat{S}^{CC}(k) \Delta^{CC}(k). \end{aligned}$$

Fig. 7. A potentially universal contribution.

Many of the remaining terms in the tree-level equation for $S_{\mu\nu}^{AACC}$ may be further simplified by making use of the relation between the integrated kernel and the corresponding two-point function. Such a relation may be easily obtained from the tree-level equation for the effective two-point coupling, in the present example S^{CC} . By rewriting it in terms of the inverse coupling, $(S^{CC})^{-1}$, we get $(S^{CC})^{-1} = \Delta^{CC}$, *i.e.* $S^{CC} \Delta^{CC} = 1$. This leads to the simplifications shown in Fig. 8.

The last step concerns how to handle the terms that contain two three-point effective couplings. The procedure is pretty much the same, except that one has to recognise the derivative

of the “square of the kernel” (see Fig. 9). At the algebra level, it amounts to writing the second diagram in Fig. 9 as the sum of two equal contributions and, then, to shifting the loop momentum so as to complete the Λ -derivative.

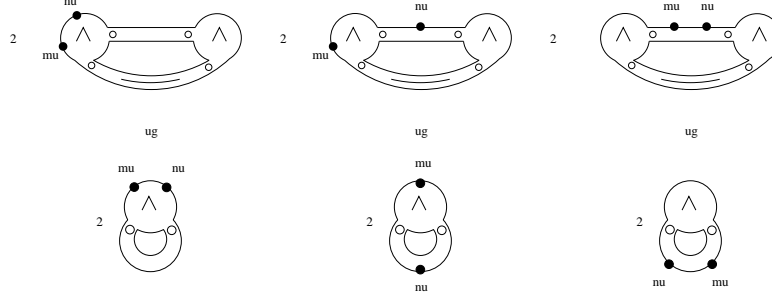


Fig. 8. Simplifications in the four-point effective vertex contribution.

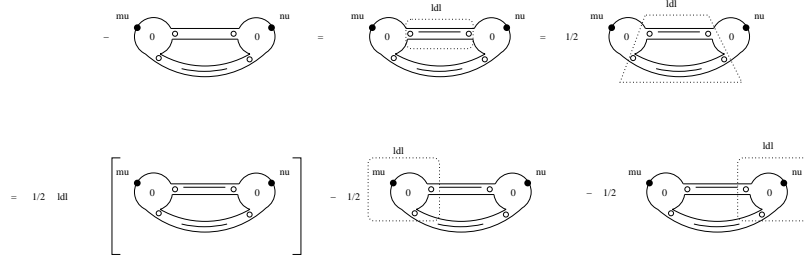


Fig. 9. How to handle two joined three-point effective vertices.

The procedure outlined in the above can be used in the whole calculation: all the hatted vertices cancel out and one is left with potentially universal terms only. The relation between integrated kernels and their corresponding two-point functions, however, is more complicated in the general case. As a matter of fact, it takes the form $S_{IK}(p)\Delta_{KJ}(p) = \delta_{IJ} + R_{IJ}(p)$, where the “remainder” R_{IJ} , absent in the scalar sector, is a (un-)broken gauge transformation. In the A sector, for example, $R_{\mu\nu}(p) = -\frac{p_\mu p_\nu}{p^2}$.

Once the potentially universal terms have been collected, the momentum integrals should be carried out. We used dimensional regularisation as a preregulator to avoid all the subtleties related to cancelling divergences against each other. (Had we done the calculation in a way that preserves $SU(N|N)$, preregularisation would not have been needed.)

5 Conclusions

A manifestly gauge invariant RG has been proposed. Together with the necessary gauge invariant regularisation [5], it constitutes a very powerful method of computation in gauge theory, as it al-

lows one to calculate the Wilsonian effective action and extract the physics in a way that respects the gauge symmetry, with no need for gauge fixing and accompanying ghosts [3, 8]. Hence the Gribov problem is completely avoided [11].

As a basic test of the formalism, the one-loop $SU(N)$ beta function has been computed and the expected universal result has been obtained. The strategy which has proven to be very efficient consists in eliminating the elements put in by hand by using the equations of motion for the effective action vertices, where physics is actually encoded. (See also [9] for the analysis of the scalar case). A diagrammatic technique to represent the various vertices has been sketched, and already at the level of diagrams the big potential of the method comes out.

The calculation is totally independent of the details put in by hand, such as the choice of covariantisation and the cutoff profile, and gauge invariance is no doubt the main ingredient all the way to the final result.

We hope the procedure is quite general and may be used to investigate non-perturbative aspects of gauge theories.

Including matter in the fundamental representation as well as space-time supersymmetry seems not too difficult, thus opening the door to many further avenues of exploration.

Acknowledgement: The authors wish to thank the organisers of the fifth international Conference “RG 2002” for providing such a stimulating environment. T.R.M. and S.A. acknowledge financial support from PPARC Rolling Grant PPA/G/O/2000/00464.

References

- [1] K. G. Wilson: *Phys. Rev. D* **10** (1974) 2445 ; *New Phenomena in Subnuclear Physics* (Erice '75), ed. A. Zichichi (Plenum Press, New York, 1977); *Recent Developments in Gauge Theories* (Cargèse '79), ed. G. 't Hooft (Plenum Press, New York, 1980).
- [2] Yu. A. Kubyshin: these Proceedings; D. F. Litim: these Proceedings; J. M. Pawłowski: these Proceedings; T. R. Morris: *New Developments in Quantum Field Theory*, NATO ASI series 366, (Plenum Press, New York, 1998); J. Berger, N. Tetradis and C. Wetterich: *Phys. Rep.* **363(4-6)** (2002) 223 ; C. Bagnuls and C. Bervillier: *Phys. Rep.* **348** (2001) 91 ; J. Polonyi: hep-th/0110026.
- [3] T. R. Morris: *Nucl. Phys. B* **573** (2000) 97 ; T. R. Morris: *J. High Energy Phys.* **0012** (2000) 012 .
- [4] J. Polchinski: *Nucl. Phys. B* **231** (1984) 269 .
- [5] Stefano Arnone, Yuri A. Kubyshin, Tim R. Morris, John F. Tighe: “Gauge invariant regularisation via $SU(N|N)$ ”, to be published in the *Int. J. Mod. Phys. A*.
- [6] S. Arnone, A. Gatti and T. R. Morris: “A manifestly gauge invariant ERG”, these Proceedings.
- [7] Jose I. Latorre and Tim R. Morris: *J. High Energy Phys.* **0011** (2000) 004 .
- [8] S. Arnone, A. Gatti and T. R. Morris: in preparation.
- [9] S. Arnone, A. Gatti and T. R. Morris: “Exact scheme independence at one loop”, hep-th/0201237.
- [10] S. Arnone, A. Gatti and T. R. Morris: “A demonstration of scheme independence in scalar exact RGs”, these Proceedings.
- [11] V. Gribov: *Nucl. Phys. B* **139** (1978) 1 ; I. Singer: *Comm. Math. Phys.* **60** (1978) 7 .